Rachel Minster

Proving the Continuum Hypothesis Independent of the Axioms of Set Theory

Set theory is the branch of mathematics dealing with infinite sets. Some infinities are bigger than others, and set theory seeks to analyze them. Both the real numbers and the integers are infinite sets, for example, but the set of real numbers is larger than the integers. We are generally interested in dealing with ω, the set of natural numbers, as it is one of the smaller infinite sets. The Zermelo-Fraenkel Axioms of set theory define the ways we can manipulate infinite sets in order to analyze them more effectively. Combined with the Axiom of Choice, used in multiple areas of mathematics, we have the ZFC Axiom System. One way we compare infinite sets is the cardinality of a set, which generally refers to its size.

Georg Cantor originally proposed the Continuum Hypothesis, and later on Kurt Gödel and Paul Cohen worked more to determine its truth. Cantor's original theorem, which can be proven, states that the power set of ω, or the set of all its subsets, is greater than or equal to the least cardinal greater than ω. The Continuum Hypothesis deals with the equality part of the theorem. If true, the hypothesis would be incredibly significant in that infinite sets greater than ω could be countable. However, Cohen showed that it is independent from the ZFC Axioms, so it cannot be proven. Gödel constructed a model of set theory called the constructible universe. In this universe, the Continuum Hypothesis does hold true. The difference here makes the Continuum Hypothesis so interesting to study.

For my thesis, I will research the historical significance of the Continuum Hypothesis, exploring the foundations of set theory to prove its inconsistency. I will also prove that it holds true in the constructible universe.