Title

Classifying Integral Matrices Using Polynomial Equations and Similarity

Abstract

Matrix algebra is one of the fastest expanding areas of mathematics. This is partly due to that fact that it’s one the youngest areas of mathematics. With this in mind, researching the murky or even completely unknown areas of matrix algebra can be very exciting and rewarding. The infinite amount of different matrix possibilities makes it seem impossible to organize or classify, but by adding certain requirements to the matrices you can drastically reduce the number of possible matrices. There are many different conditions you can use to organize the matrices, one of which being the determinant of the matrices. The use of this condition hasn’t been widely published or studied, which gives scholars the chance to explore and document it.

Statement of the Intent

The goal of this project is to classify 3x3 integral matrices with determinants equal to two or negative two, through matrix similarity. To do this we will use the characteristic equations of all the matrices. The end goal is to find approximately 14 unique characteristic polynomials with every integral matrix, whose determinant is equal to negative two or two, having one of these unique polynomials as its characteristic equation. This will allow us to classify the different matrices through matrix similarity, by using the characteristic polynomials
they have. My thesis advisor is Dr. Xingde Dai, who has guided me along this far and will continue doing so throughout the project.

**Significance**

Once this project is finished, we hope to prove that you can classify integral matrices by the matrices that they’re similar to. This type of classification of matrices hasn’t been documented very well, so this project is meant to explore this process and its results. By doing so, we hope to shed some light on this seemingly unknown area of matrix algebra. Hopefully this will inspire more people to research this topic and even try to recreate it by using different determinant values. Further study into this topic could lead to the discovery of different uses each of these classifications could uniquely have.

**Previous Research**

There has been lots of research done on matrix conditions based on different matrix structures and different multiplicative properties that present in matrices. Multiple different types of matrices have been defined, including diagonal matrices, orthogonal matrices, upper right triangular matrices, lower right triangular matrices, symmetric matrices, idempotent matrices, invertible matrices, and many more. These classifications are heavily used in matrix transformations, where their classification helps in the multiplication of matrices, allowing a single matrix to be written as the multiplication of other matrices. The following terms bellow are different conditions or values found in matrices that will be significant in this project.
• Eigenvalues and Eigenvectors: an eigenvector is a non-scalar vector (V) that when multiplied with the corresponding matrix (A), equals the product of the eigenvector and the corresponding scalar eigenvalue (\(\lambda\)). This is generally written as \(AV = \lambda V\).

• Characteristic Equation: the characteristic equation of a square matrix A is shown by the equation, \(\det(A-\lambda I) = 0\), where \(\lambda\) is a diagonal matrix composed of matrix A’s eigenvalues and \(I\) is the identity matrix.

  o This equation can be rewritten as \(\lambda^3 + a\lambda^2 + b\lambda + c = 0\), where \(a\), \(b\), and \(c\) are all integers that can be calculated using matrix A’s eigenvalues.

  o Hence \(a = - (\lambda_1 + \lambda_2 + \lambda_3)\), \(b = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1\), and \(c = \lambda_1\lambda_2\lambda_3 = \det(A)\), with \(\lambda_1\), \(\lambda_2\), and \(\lambda_3\) all being the three eigenvalues of matrix A.

  o One important note that cannot be skipped is that as \(c = \det(A)\). Using that fact along with the fact that the matrices are filled with only integers, allows you to approximate the bounds of the possible eigenvalues of matrix A (\(\lambda_1\), \(\lambda_2\), and \(\lambda_3\)). Hence allowing you to further approximate the values of \(a\) and \(b\). This means that by choosing to work only with integral matrices that all have the same desired determinant value, you can now organize that down to a finite number of characteristic polynomials.

• Minimal Polynomial: the minimal polynomial of a matrix A is the monic (the coefficient for the highest degree term is equal to 1) polynomial of the smallest degree \(n\) such that

\[
p(A) = \sum_{i=0}^{n} c_i A^i = 0.
\]
The minimal polynomial divides the characteristic equation. Hence if the characteristic equation factors as
\[ \text{char}(A)(x) = (x - \lambda_1)^{m_1} \cdots (x - \lambda_k)^{m_k}, \]
then the minimal polynomial is
\[ p(x) = (x - \lambda_1)^{m_1} \cdots (x - \lambda_k)^{m_k} \]
with \( 1 \leq m_i \leq n_i \). 

- **Similar Matrices:** Matrix A is similar to matrix B if the following two conditions are met.
  - If \( \text{abs}(A - \lambda I) = \text{abs}(B - \lambda I) \)
  - And if the minimum polynomial of A equals the minimum polynomial of B

**Proposed Method**

This project will be completed by using mathematical theory, applying matrix algebra, and using computer programming in Matlab. The steps general steps that will be taken are shown below.

1. Choose a determinate value to work with, and we chose to set the determinate value equal to 2 or -2.
2. From there we will approximate bounds for the eigenvalues, which leads us to approximate the possible values of a and b for the characteristic equation.
3. Once a and b have appropriate bounds, I can start programming a function that will solve each possible characteristic equation and store the corresponding eigenvalues.
4. Then we will need to eliminate characteristic equations that wouldn’t be possible to attain through integral matrices with determinant values of two or negative two. To do
this we will have to apply matrix algebra theory to set standards for the appropriate
eigenvalues and potentially work with the problem of sets of eigenvalues that contain
multiple of the same value. We will also have to check whether each characteristic
equation can be found in two non-similar matrices.

5. Once we have successfully analyzed all of the characteristic equations and their
   corresponding eigenvalues, we will then try to generate integral matrices (with
determinates of two or negative two) whose characteristic polynomials are equal to
each of the fourteen that we find.

**Conclusion**

Once all of the steps have been completed, Dr. Dai expects that we will find 14 unique
characteristic polynomials. We also expect to be able to generate some similar matrices with
the same characteristic polynomial. With all of this completed and documented, it could be a
vital resource to further research of topics like this and will hopefully inspire more people to
conduct said research.
Bibliography


